Effect of the Subdivision Strategy on Convergence and Efficiency of Some Global Optimization Algorithms

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Abstract. We investigate subdivision strategies that can improve the convergence and efficiency of some branch and bound algorithms of global optimization. In particular, a general class of so called weakly exhaustive simplicial subdivision processes is introduced that subsumes all previously known radial exhaustive processes. This result provides the basis for constructing flexible subdivision strategies that can be adapted to take advantage of various problem conditions.

Key words. Branch and bound, global optimization, subdivision strategy, exhaustive and weakly exhaustive subdivision processes.

1. Introduction

Many global optimization algorithms of branch and bound type involve a subdivision process which can be described by a tree. The root of this tree represents an initial simplex S_0 and a path in the tree is a nested sequence of simplices $S_0 \supset \cdots \supset S_k \supset \cdots$ such that S_{k+1} is obtained via the radial subdivision of S_k with respect to some point $w^k \in S_k$ (for the definition of radial subdivision, cf. [5]).

Most often, the convergence and efficiency of the algorithm critically depend upon the subdivision strategy, i.e. the way of choosing the subdivision point w^k. A fundamental question that arises here is under which conditions one has

$$\delta(S_k) \rightarrow 0? \tag{1}$$

where $\delta(S)$ denotes the diameter, i.e. the length of the longest edge, of S. If this holds for every nested sequence generated by the subdivision process we say that the process is *exhaustive*. The interest of this concept stems from the fact that for many algorithms using certain standard bounding methods the consistency of the bounding operation and hence the convergence of the algorithm will be guaranteed if the underlying subdivision process is exhaustive (see [9]).

The most commonly used exhaustive subdivision process is the *bisection* in which each simplex is divided into two subsimplices by a hyperplane passing through the midpoint of a longest edge and all the vertices that are not incident to this edge (cf. [3] and [5]). Unfortunately, it has been observed that the convergence of algorithms based on bisection is too slow (cf. e.g. [2]). This deficiency is due to two reasons: 1) at each bisection the volume of a simplex diminishes just by a half, so many bisections may be needed to get a sufficiently small subsimplex; 2) the bisection is a "passive" strategy which does not take the problem conditions into account. Therefore, other exhaustive subdivision processes have been introduced (cf. [2], [11], [12] and recently [8]). Computational experiments reported in [2], [11], [12], [4] have effectively confirmed the superiority of these flexible subdivision strategies compared to the bisection.

However, even when realized by a flexible subdivision strategy, exhaustiveness is too strong a condition which may be costly to achieve in order to secure the convergence of a given algorithm. The aim of this paper is to present a weaker condition that, while sufficient for convergence, is much easier to realize practically. Both from the computational and theoretical point of view, the subdivision processes realizing this condition are far more satisfactory than all exhaustive subdivision processes previously known.

In the next section, we establish a very general condition of exhaustiveness which subsumes all the conditions earlier developed in [11] and [12]. On the basis of this general result, Section 3 introduces the concept of weak exhaustiveness which allows a great flexibility in realizing subdivision strategies able to accommodate convergence with efficiency. Section 4 discusses the applications to conical and simplicial algorithms, providing substantial improvements. Finally, Section 5 discusses some peculiarities of rectangular subdivisions, which are suited for separable problems.

2. A General Condition for Exhaustiveness

Let $S = [s^1, \ldots, s^n]$ be an $(n-1)$-simplex in R^n and let w be an arbitrary point of S:

$$w = \Sigma \lambda_i s^i , \quad \lambda_i \geqq 0 , \quad \Sigma \lambda_i = 1 ,$$

where the set $J = \{i : \lambda_i > 0\}$ has at least two elements (i.e. $w \neq s^i \, \forall\, i$). For each $j \in J$ form the simplex $S(j, w)$ whose vertex set is obtained from that of S by replacing s^j with w. Then the collection $\{S(j, w) : j \in J\}$ constitutes a partition of S (see, e.g., [5]) which is called a *radial subdivision* of S with respect to w. Clearly, the bisection corresponds to the case where w is the midpoint of a longest edge of S.

Now consider an infinite nested sequence of simplices

$$S_1 \supset S_2 \supset \cdots \supset S_k \supset \cdots \tag{2}$$

such that S_{k+1} is obtained from S_k via a radial subdivision with respect to some $w^k \in S_k$. The question that arises is under which conditions this sequence is exhaustive, i.e. satisfies (1)?

For each $i = 1, \ldots, n$ denote

$$\delta(i, S) = \max\{\|s^i - s^j\| : j \neq i\} ,$$

i.e., $\delta(i, S)$ is the maximal length of an edge of S incident to s^i.

DEFINITION. Given an $(n-1)$-simplex $S=[s^1, \ldots, s^n]$ and a constant $\rho \in (0,1)$ we say that a point $w \in S$ satisfies the *ρ-eccentricity condition* in S if

$$\max\{\|w - s^i\|; i = 1, \ldots, n\} < \rho\delta(S) .$$

A point $w \in S$ satisfies the *ρ-dominance condition* in S if

$$w \in \operatorname{conv}\{s^i : \delta(i, S) > \rho\delta(S)\} .$$

Let $S_k = [s^{k1}, \ldots, s^{kn}]$, where it is agreed that when S_{k+1} is obtained from S_k by replacing s^{ki} with w^k, then the new vertex of S_{k+1} receives the index i of the vertex of S_k that it replaces.

THEOREM 1. *If there exists a constant $\rho \in (0,1)$ such that w^k satisfies the ρ-eccentricity condition in S_k for* every k, *and satisfies the ρ-dominance condition in S_k for* infinitely many k, *then (1) holds.*

Proof. For simplicity denote $\delta_k = \delta(S_k)$. Let $k_1 < k_2 < \cdots$ be the infinite subsequence formed by all k for which u^k satisfies the ρ-dominance condition in S_k. We first show that, given any t, there exists $k > t$ such that

$$\delta_k \leqq \rho\delta_t . \tag{3}$$

Since $0 < \rho < 1$, it will then easily follow that $\delta_k \downarrow 0$. Colour every vertex of S_t "black" and colour "white" every vertex of a S_k with $k > t$ which is not black. Clearly, if a vertex s^{ki} of a simplex S_k is white then $s^{ki} = w^h$ for some h, $t \leqq h < k$, so that according to the ρ-eccentricity condition:

$$\begin{aligned} \delta(i, S_k) &= \max\{\|w^h - s^{kj}\| : j \neq i\} \\ &\leqq \max\{\|w^h - s^{hj}\| : j \neq i\} \leqq \rho\delta_h \leqq \rho\delta_t . \end{aligned} \tag{4}$$

Therefore, if for some $k > t$ the simplex S_k has a white vertex incident to a longest edge e_k, then $\delta_k = \|e_k\| \leqq \rho\delta_t$ and (3) holds. Suppose now that for all $k > t$, a longest edge e_k of S_k has always two black endpoints, i.e. is an edge of S_t. Then, since the number of edges of S_t is finite, there is a fixed edge e of S_t such that $e_k = e$ for infinitely many k, hence $\delta_k = \|e_k\| = \|e\|$ for all sufficiently large k, say for all $k \geqq t'$. Without loss of generality we can assume $t' = t$, so that $\delta_k = \delta_t$ for all $k \geqq t$. Also we can assume that $k_1 > t$. By hypothesis the ρ-dominance condition is satisfied for $k = k_1$. But, according to (4), any white vertex s^{ki} of S_k satisfies $\delta(i, S_k) \leqq \rho\delta_h = \rho\delta_k$, so that any vertex s^{ki} of S_k with $\delta(i, S_k) > \rho\delta_k$ must be black. Thus, for $k = k_1$, w^k belongs to a face F_k of S_k spanned by black vertices. Since S_{k_1+1} is obtained from S_{k_1} by replacing a vertex of F_{k_1} by w^{k_1}, it follows that S_{k_1+1} has at least one black vertex less than S_{k_1}. Similarly, S_{k_2+1} has at least one black vertex less than S_{k_2}. And so on. Continuing in this way we will arrive at a t_1 such that S_{t_1} has no black vertex. This contradicts the assumption that any longest edge of S_k for $k > t$ has black endpoints. Therefore, there exists $k > t$ such that (3) holds. □

REMARK. The Theorem still holds if instead of the ρ-eccentricity condition we only assume

$$\max\{\|w^k - s^{kj}\| : j \neq i_k\} \leqq \rho\delta_k , \tag{5}$$

where i_k is the index of the vertex which is to be replaced by w^k. Actually, only (5) was used in the above proof.

COROLLARY 1. *The bisection process is exhaustive.*

Proof. For each simplex $S = [s^1, \dots, s^n]$ denote by $v(s)$ the midpoint of a longest edge of S. It is known that $v(S)$ satisfies the ρ-eccentricity condition in S with $\rho = \sqrt{3}/2$ (cf. [8] or [5]). Furthermore, $v(S)$ obviously satisfies the ρ-dominance condition for any $\rho \in (0, 1)$. Therefore, if $w^k = v(S_k)$ for every k (i.e. if S_{k+1} is always obtained from S_k via a bisection) then the conditions of Theorem 1 are fulfilled by any infinite nested sequence $\{S_k\}$ generated by the subdivision process. □

COROLLARY 2. *If there exists a constant* $\rho \in (0, 1)$ *such that for every* k*:*

(a) $max\{\|w^k - s^{ki}\| : i \in J_k\} \leqq \rho\delta(S_k)$,

(b) $w^k \in conv\{s^{ki} : i \in J_k\}$,

where $J_k = \{i : \delta(i, S_k) > \rho\delta(S_k)\}$, *then* $\delta(S_k) \to 0$ *as* $k \to \infty$.

Proof. Condition (a) implies that $\max\{\|w^k - s^{ki}\| : i = 1, \dots, n\} \leqq \rho\delta(S_k)$, since for $i \notin J_k$ one has $\|w^k - s^{ki}\| \leqq \delta(i, S_k) \leqq \rho\delta(S_k)$. On the other hand (b) means that w^k satisfies the ρ-dominance condition in S_k. Therefore, the conditions of Theorem 1 are fulfilled (with the ρ-dominance condition holding for *all* and not only for infinitely many k). □

A variant of this "balanced subdivision method" is to choose the subdivision point $w(S)$ for each simplex $S = [s^1, \dots, s^n]$ in such a way that for some constant $\rho \in (0, 1)$:

$$w(S) = \Sigma\lambda_i s^i \text{ with } \lambda_i \geqq 0 , \quad \Sigma\lambda_i = 1$$
$$\lambda_i > 0 \Leftrightarrow \delta(i, S) > \rho\delta(S) ; \quad \min\{\lambda_i : \lambda_i > 0\} \geqq 1 - \rho .$$

Indeed, it can be verified that the last inequality implies that $\|w(S) - s^i\| \leqq \rho\delta(S)$ for all i with $\lambda_i > 0$.

Although the balanced method performs better than the bisection (cf. [11]), it is still determined beforehand and does not vary adaptively in order to take advantage of the information gathered as the algorithm proceeds.

3. Weakly Exhaustive Process

One may wonder whether one of the conditions required in Theorem 1 can be dropped:

(i) ρ-eccentricity for all k;

(ii) ρ-dominance for infinitely many k.

Clearly, a sequence $\{S_k\}$ may satisfy (i) while $\lim \delta(S_k) > 0$. For example, this is the case if for every k, w^k = centre of gravity of S_k and $s^{k+1,1} = w^k$. The following counter example shows that condition (ii) alone is not sufficient either.

Take any decreasing sequence of real numbers $\alpha_{2k} \downarrow 1$ and in R^2 construct an infinite nested sequence of triangles (S_k) as follows. Let S_0 be any triangle such that $\|b_0\| > \alpha_0$ (b_k denotes the second longest edge of S_k). Bisect S_0 and let S_1 be the subtriangle of S_0 that contains b_0. Then as w^1 (the subdivision point of S_1) choose a point lying on an edge of S_1 other than b_0 but so near to an endpoint of b_0 that w^1 determines with b_0 a subtriangle of S_1 having at least two edges of length $\geqq \alpha_0 > \alpha_2$. Let S_2 be this subtriangle of S_1. Then, since $\|b_2\| > \alpha_2$, the same process can be repeated with S_2 in place of S_0. Clearly the sequence $\{S_k\}$ so constructed will have $\delta(S_k) \geqq 1\ \forall k$, despite the fact that it involves infinitely many bisections (every S_{2h}, $h = 0, 1, \ldots$, is bisected).

Thus, even if, for infinitely many k, w^k is the midpoint of a longest edge of S_k, it is not guaranteed that $\lim \delta(S_k) = 0$.

However, from Theorem 1 we can derive the following proposition which is of fundamental importance for our purpose.

THEOREM 2. *If the sequence $\{S_k\}$ involves infinitely many bisections (i.e. for infinitely many k, say $k \in \Delta$, w^k is the midpoint of a longest edge of S_k), then there exists a subsequence $\{k_\nu\} \subset \{1, 2, \ldots\} \backslash \Delta$ such that, as $\nu \to \infty$, we have:*

$$w^{k_\nu} \to w,\ s^{k_\nu i} \to s^i\ (i = 1, \ldots, n),\ w \in \mathit{vert}[s^1, \ldots, s^n], \tag{6}$$

($\mathit{vert}[s^1, \ldots, s^n]$ denote the vertex set of the polytope which is the convex hull of $\{s^1, \ldots, s^n\}$).

Proof. From the hypothesis it can easily be seen that for every $k \in \Delta$ the ρ-dominance condition holds for arbitrary $\rho \in (0, 1)$ and the ρ-eccentricity condition holds for $\rho = \sqrt{3}/2$. Therefore, if for some $\rho \in (0, 1)$ the ρ-eccentricity condition holds for all sufficiently large $k \notin \Delta$, then by Theorem 1, $\delta(S_k) \to 0$ and (6) holds *a fortiori*. Consider now the case when for every $\rho \in (0, 1)$ there are infinitely many $k \notin \Delta$ for which the ρ-eccentricity condition does not hold. That is, for every ν there exists $k_\nu \notin \Delta$, such that $k_\nu > \nu$ and

$$\max\{\|w^{k_\nu} - s^{k_\nu i}\| : i = 1, \ldots, n\} > (1 - 1/\nu)\delta(S_{k_\nu}) \tag{7}$$

By taking a subsequence if necessary, we can assume that $w^{k_\nu} \to w$, $s^{k_\nu i} \to s^i$ $(i = 1, \ldots, n)$, and $\delta(S_{k_\nu}) \to \delta = \max\{\|s^i - s^j\| : i < j\}$. Then from (7) we have

$$\max\{\|w - s^i\| : i = 1, \ldots, n\} = \delta$$

and, since $w \in \operatorname{conv}\{s^1, \ldots, s^n\}$, this implies that w is a vertex of $\operatorname{conv}\{s^1, \ldots, s^n\}$. □

A sequence $\{S_k\}$ for which there exists a subsequence $\{S_{k_\nu}\}$ satisfying (6) is said to be *weakly exhaustive* and a subdivision process such that any infinite nested sequence generated by it is weakly exhaustive is called a *weakly exhaustive process*. Thus, Theorem 2 says that a subdivision process is weakly exhaustive if every infinite nested sequence that it generates involves infinitely many bisections.

It turns out that for many branch and bound procedures of global optimization, weak exhaustiveness of the subdivision process is sufficient to guarantee the convergence.

4. Applications

A global optimization algorithm usually alternates between two phases: a *local* phase in which one seeks to improve the current best solution by using relatively inexpensive local methods, and a global phase in which more expensive *global* methods are called for to test the current best solution for global optimality and if the test fails, to find a better feasible solution. As shown in [10] (cf. also [5]), for a wide class of global optimization problems including concave minimization, reverse convex programming, d.c. programming and even Lipschitz and continuous optimization problems, the global phase reduces to solving a problem of the following form:

(DC) Given in R^n a polytope D and a compact convex set C, check whether $D \subset C$, and if not, find a point of $D \setminus C$.

For example, for the problem of globally minimizing a concave function $f(x)$ over a polytope D, the global phase amounts to solving a problem (DC) with D the given polytope and $C = \{f(x) \geqq f(x^0)\}$, where x^0 is the current best solution to be tested for global optimality or transcended.

I. CONICAL ALGORITHMS

Assume that 0 is a vertex of the polytope D and there is an $(n-1)$-simplex S_0 such that: $0 \notin \operatorname{aff} S_0$, $\operatorname{conv}\{0, S_0\} \subset C$, while $D \subset \operatorname{con} S_0$ (conv A denotes the convex hull of A, con A denotes the cone generated by A). These are mild assumptions which usually can be made to hold after some simple manipulations.

A conical algorithm for solving (DC) can be outlined as follows ([8], cf. also [5]):

(To simplify the language by "C-extension of x" we mean the intersection of the boundary ∂C of C with the ray from 0 through x, for $x \neq 0$)

1) Let $\mathcal{P}_0 = \mathcal{M}_0 = \{S_0\}$. Set $k = 0$.

2) For each $S = [s^1, \ldots, s^n] \in \mathcal{P}_k$ compute the C-extensions $z^i = \theta_i s^i$ of s^i $(i = 1, \ldots, n)$ and solve the linear program

$$\mathrm{LP}(S) \quad \max \phi_s(x) \text{ s.t. } x \in D \cap \operatorname{con} S$$

where $\phi_s(x)$ is the linear function such that the hyperplane through $z^1, \ldots, z^n$ is described by the equation $\phi_s(x) = 1$.

(LP(S) amounts to maximizing $\Sigma\lambda_i/\theta_i$ s.t. $\Sigma\lambda_i s^i \in D$, $\lambda_i \geqq 0\ \forall i$).

Let $x(S)$, $\mu(S)$ denote an optimal solution and the optimal value of LP(S). If for some S, $x(S) \notin C$, then terminate; otherwise, $x(S) \in C$ for all $S \in \mathcal{P}_k$, then go to 3).

3) In $\mathcal{M}_k$ delete all $S \in \mathcal{P}_k$ such that $\mu(S) \leqq 1$. Let $\mathcal{R}_k$ be the collection of remaining simplices. If $\mathcal{R}_k = \emptyset$ then terminate: $D \subset C$. Otherwise, go to 4).

4) Select $S_k \in \operatorname{argmax}\{\mu(S) : S \in \mathcal{R}_k\}$ and subdivide S_k with respect to some point $w^k \in S_k$.

5) In $\mathcal{R}_k$ replace S_k by its partition $\mathcal{P}_{k+1}$ and let $\mathcal{M}_{k+1}$ be the resulting collection of simplices. Set $k \leftarrow k+1$ and return to 1).

Clearly the simplicial subdivision process performed on S_0 induces a conical subdivision process on con $S_0 \supset D$ (hence the name "conical algorithm").

Convergence and efficiency of the above algorithm depend upon the subdivision strategy, i.e. the concrete rule for choosing the subdivision point w^k in Step 4.

It can be proved (cf. [7]) that if each subdivision is a bisection (i.e. if always w^k = midpoint of a longest edge of S_k), then convergence is guaranteed in the following sense:

If $D \backslash C \neq \emptyset$ or if $D \subset \operatorname{int} C$ then the algorithm terminates after finitely many steps (yielding a point of $D \backslash C$ or establishing that $D \subset C$, respectively).

Unfortunately, as evidenced by computational experiments, convergence with the bisection process is very slow. On the other hand, if we always choose $w^k = \omega^k :=$ intersection of the ray from 0 through $x(S_k)$ with S_k, then convergence is not guaranteed (jamming is possible) but in many cases the algorithm works quite well. Thus, there is some conflict between convergence and efficiency, as far as the subdivision strategy is concerned.

Following [8] (cf. also [5]) we shall refer to a subdivision process in which $w^k = \omega^k \; \forall k$ as an *ω-subdivision process.* A subdivision process is said to be *normal* if every infinite nested sequence $(S_k, k \in \Gamma\}$ generated by this process satisfies

$$\lim \mu(S_k) = 1 (k \to \infty, k \in \Gamma) . \tag{8}$$

It has been proved in [5] that convergence (in the same sense as above) is guaranteed when the subdivision process is normal (it is easily seen that the normality condition defined here is in fact equivalent to that given in [5]). Using this result from [5] and Theorem 2 of Section 3 we now show that the mentioned conflict can be resolved by "normalizing" the ω-subdivision process, i.e. by constructing a normal subdivision process which does not differ much from the ω-subdivision process.

Denote by $\tau(S)$ the generation index of S, which is computed by setting $\tau(S_0) = 1$ and $\tau(S') = \tau(S) + 1$ whenever S' is a "son" of S. Select a natural N (typically $N \geqq 5$) and a sequence $\alpha_k \downarrow 0$.

NORMALIZED ω-SUBDIVISION (NωS) RULE. If $\tau(S_k)$ is a multiple of N and $\mu(S_k) > 1 + \alpha_k$ then perform a bisection of S_k; otherwise, divide S_k with respect to $w^k = \omega^k$.

THEOREM 3. *The $N\omega S$ Rule generates a normal subdivision process.*

Proof. Consider any infinite nested sequence $\{S_k, k \in \Gamma\}$ generated by the algorithm. Let $\Delta = \{k \in \Gamma : S_k \text{ is bisected}\}$. If Δ is finite then (8) is obvious, since $\mu(S_k) \leqq 1 + \alpha_k$ for all sufficiently large k such that $\tau(S_k)$ is a multiple of N. Suppose now that Δ is infinite. Let q^k, u^k denote the intersections of the ray from 0 through $x(S_k)$ with $[z^{k1}, \ldots, z^{kn}]$ and ∂C, respectively. Since $\mu(S_k) = \|x(S_k)\| / \|q^k\|$, $x(S_k) \in [u^k, q^k]$, and $\|q^k\|$ is bounded below, (8) will be proved if we show that

$$\lim \|q^k - u^k\| = 0 \ (k \to \infty,\ k \in \Gamma \backslash \Delta) .$$

In view of Theorem 2 and the compactness of S_0 and ∂C, without loss of generality we can assume that, as $k \to \infty$ $(k \in \Gamma \backslash \Delta)$:

$$s^{ki} \to s^i (i = 1, \ldots, n),\ \omega^k \to s^1 \in \operatorname{vert}[s^1, \ldots, s^n] ,$$
$$z^{ki} \to z^i = \theta_i s^i \ (i = 1, \ldots, n) .$$

Now, observe that if $\pi(x)$ denotes the C-extension of $x \in S_0$ then there is a constant $\eta > 0$ such that $\|\pi(x') - x(x'')\| \leqq \eta \|x' - x''\|$ for all $x', x'' \in S_0$. Indeed, clearly $\pi(x) = x/p(x)$, where

$$p(x) = \inf\{\lambda \geqq 0 : x \in \lambda C\}$$

is the gauge of C. Since $p(x)$ is convex, hence Lipschitz over S_0, and $p(x)$ is bounded below over S_0, it easily follows that $\pi(x)$ is also Lipschitz over S_0. Therefore,

$$\begin{aligned} \|u^k - z^{k1}\| &= \|\pi(w^k) - \pi(s^{k1})\| \\ &\leqq \eta \|w^k - s^{k1}\| \to 0 \ (k \to \infty,\ k \in \Gamma \backslash \Delta) . \end{aligned} \tag{9}$$

On the other hand, obviously $q^k = \alpha_k \omega^k$ for some $\alpha_k \geqq 1$, so if $q = \lim q^k$ then $q = \alpha s^1$ for some $\alpha \geqq 1$. But from $q^k \in [z^{k1}, \ldots, z^{kn}]$ we have $q \in \operatorname{conv}(z^1, \ldots, z^n\}$, i.e. $q = \Sigma_i \zeta_i z^i = \Sigma_i \zeta_i \theta_i s^i$, hence $\alpha s^1 = \Sigma_i \zeta_i \theta_i s^i$ and setting $\beta_i = \xi_i \theta_i / \alpha$, we obtain

$$s^1 = \sum_i \beta_i s^i$$

with $\beta_i \geqq 0$, $\Sigma_i \beta_i = 1$ (this is because $0 \notin \operatorname{aff} S_0$ implies that $s^1 \in S_0$ only if $\Sigma_i \beta_i = 1$). Since $s^1 \in \operatorname{vert}[s^1, \ldots, s^n]$ we must then have $\beta_i = 0$ for $i \notin I := \{i : s^i = s^1\}$, i.e. $\Sigma_{i \in I} \beta_i = 1$. Noting that $\theta_i \geqq 1 \, \forall i$ and $\theta_i = \theta_1$ $(i \in I)$ this yields $\zeta_i = 0$ $(i \notin I)$ and $\Sigma_{i \in I} \beta_i = \Sigma_{i=1}^n \zeta_i \theta_1 / \alpha = \theta_1 / \alpha = 1$, i.e. $\alpha = \theta_1$, hence $q = \theta_1 s^1 = z^1$. Thus,

$$\|q^k - z^{k1}\| \to \|q - z^1\| = 0 \ (k \to \infty,\ k \in \Gamma \backslash \Delta) . \tag{10}$$

From (9) and (10) we conclude, as was to be proved,

$$\|q^k - u^k\| \leqq \|q^k - z^{k1}\| + \|z^{k1} - u^k\| \to 0 \ (k \to \infty,\ k \in \Gamma \backslash \Delta) . \qquad \square$$

Thus, to ensure the convergence of Algorithm 1 we need only weakly exhaustive subdivision processes.

REMARKS. (i) In [8] an exhaustive process was proposed that could be derived from Theorem 1, namely:

If $\tau(S_k)$ is a multiple of N or the ρ-eccentricity condition is not satisfied ($\rho \in (\sqrt{3}/2, 1)$ being a user supplied parameter) then perform a bisection of S_k. Otherwise, divide S_k with respect to $w^k = \omega^k$.

Since checking the ρ-eccentricity condition may be time consuming (and besides, this condition is likely to hold in most cases when ρ is very close to 1), in practice this rule is often used in the following loose form:

Choose $w^k = \omega^k$ as long as the algorithm proceeds normally and use a bisection only when the algorithm is slowing down.

Although this loose rule does not guarantee normality, computational experiments have shown that it works quite well [4] (any way much better than the pure bisection rule). In light of the above results, this loose rule can now be given a precise formulation:

Choose $w^k = \omega^k$ as long as $\mu(S_k) \leqq 1 + \alpha_k$ and use a bisection only when $\mu(S_k) > 1 + \alpha_k$. (In fact, the speed of convergence of the algorithm can be evaluated from the speed of convergence of the quantity $\mu(S_k) - 1$ to zero).

Since this is a special realization of the $N\omega S$ Rule, normality (and hence, convergence) is assured.

(ii) The above algorithm still works when C is unbounded, provided D remains bounded (then we agree that $1/\theta_i = 0$ if $\theta_i = +\infty$). When D itself is unbounded, an extension of the algorithm requiring an exhaustive subdivision has been proposed in [10].

II. SIMPLICIAL ALGORITHMS

The rationale for using conical subdivision is that if the set $D \backslash C$ is nonempty, at least one point of it lies on the boundary of D, so that the search for such a point can be concentrated on this boundary. However , there are instances where other subdivision methods might be preferred.

As an example consider the problem (DC) when C has the form

$$C = \{(y, z) \in R^p \times R^q : g(y) \leqq h(z)\} ,$$

where $p + q = n$, $g: R^p \to R$ is a convex function, while $h: R^q \to R$ is an affine function. Since only the y-variables enter the problem in a nonlinear way, it is more convenient, when solving the problem by branch and bound, to branch with respect to the y-variables. Let S_0 be a p-simplex in $Y = R^p$ which contains the projection of D on Y. Then the problem is to find a point $y \in S_0$ for which there is z satisfying $(y, z) \in D$ and $g(y) - h(z) > 0$.

A simplicial algorithm for this problem is similar to the conical algorithm, but the space is partitioned into subsets of the form $S \times R^q$, where $S = [s^1, \ldots, s^{p+1}]$ is a subsimplex of S_0, and for each such subset we consider the linear program:

$$\begin{aligned} \mathrm{LP}(S) \quad & \max \ [\Sigma \lambda_i g(s^i) - h(z)] \\ & \text{s.t. } (\Sigma \lambda_i s^i, z) \in D, \Sigma \lambda_i = 1, \lambda_i \geqq 0 \, \forall i . \end{aligned}$$

If $\gamma(S)$ is the optimal value of LP(S), then S is deleted if $\gamma(S) \leqq 0$, while S is chosen for branching if it has maximal $\gamma(S)$ among all simplices still of interest at the given stage. The algorithm terminates when some LP(S) has an optimal solution (λ, z) such that $g(\Sigma\lambda_i s^i) - h(z) > 0$ (then a solution of (DC) is obtained) or when no simplex remains for consideration (then $D \subset C$).

It can be proved that the algorithm will converge if the subdivision process is *normal* in the following sense: for every infinite nested sequence $\{S_k, k \in \Gamma\}$ generated by the process, such that (λ^k, z^k) is a basic optimal solution of LP(S_k) and $y^k = \Sigma\lambda_i^k s^{ki}$, where s^{ki} are the vertices of S_k, we have

$$\lim \left[\sum_i \lambda_i^k g(s^{ki}) - g(y^k)\right] \to 0 \ (k \to \infty, k \in \Gamma)\,.$$

Just as with the conical algorithm, the following $N\omega S$ rule will ensure normality, hence convergence, of the subdivision process (the proof is similar to that of Theorem 3):

Select a natural N and a sequence $\alpha_k \downarrow 0$.

If $\tau(S_k)$ is a multiple of N and $\Sigma\lambda_i^k g(s^{ki}) - g(y^k) > \alpha_k$ then bisect S_k; otherwise, divide S_k with respect to y^k.

5. Separable Problems and Rectangular Algorithms

For certain problems rectangular subdivisions may be more appropriate than conical or simplicial subdivisions.

A rectangle $M = [r, s] = \{x : r \leqq x \leqq s\}$ is the cartesian product of n intervals $M_j = [r_j, s_j]$. What makes the interest of rectangular subdivisions for our purpose is the fact that for a *separable* function $f(x) = \Sigma f_j(x_j)$, there is on each rectangle $M = [r, s]$ a unique affine function $\phi_M(x)$ that agrees with $f(x)$ at the vertices of M (namely the function $\phi_M(x) = \Sigma\phi_{Mj}(x_j)$, where $\phi_{Mj}(t)$ is the affine function of one variable that agrees with $f_j(t)$ at the endpoints of $[r_j, s_j]$).

Given a point $w \in M$ and a nonempty set $J \subset \{1, 2, \dots, n\}$ we can consider the subdivision of the rectangle M into subrectangles of the form $\Pi_{j=1}^n P_j$, where

$$P_j = [r_j, s_j] \text{ if } j \notin J \text{ and}$$
$$P_j = [r_j, w_j] \text{ or } [w_j, s_j] \text{ if } j \in J\,.$$

This subdivision will be referred to as a subdivision via (w, J). Below we shall only consider rectangular subdivisions of this type.

Let $M_1 \supset M_2 \supset \cdots \supset M_k \supset \cdots$ be an infinite nested sequence of rectangles such that M_{k+1} is obtained from M_k by a subdivision via (w^k, J_k). A nice property of rectangular subdivision processes is their weak exhaustiveness, independently of the choice of the (w^k, J_k). Their property is induced by the same property of simplicial subdivision processes in one-dimensional space.

Denote $\eta_{kj} = \min(|w_j^k - r_j^k|, |w_j^k - s_j^k|)$.

THEOREM 4. *In any rectangular subdivision process, every infinite nested sequence* $\{M_k, k \in \Gamma\}$ *satisfies*

$$\underline{\lim} \max\{\eta_{kj} : j \in J_k\} = 0 \ (k \to \infty, k \in \Gamma) . \tag{11}$$

Proof. Since $J_k \subset \{1, \ldots, n\}$ there is an infinite subsequence $\Delta \subset \Gamma$ such that $J_k = J \ \forall k \in \Delta$. For any fixed $j \in J$ denote $\delta_{kj} = |s_j^k - r_j^k|$. Either of the following alternatives holds: (1) there is a constant $\rho \in (0, 1)$ such that $\eta_{kj} > \rho\delta_{kj}$ for all sufficiently large $k \in \Delta$; 2) there is an infinite subsequence $\delta_j \subset \Delta$ such that for $k \in \delta_j : \eta_{kj} \leqq \rho_k \delta_{kj}$, where $\rho_k \downarrow 0$. In the first case $\delta_{kj} \leqq (1-\rho)\delta_{hj}$ for $h < k$ ($h \in \Delta$), hence $\eta_{kj} \leqq \delta_{kj} \to 0$ ($k \to \infty, k \in \Delta$); in the second case, obviously $\eta_{kj} \to 0$ ($k \to \infty, k \in \Delta_j$). Thus, for each $j \in J$ there is an infinite sequence $\Delta_j \subset \Delta$ such that $\eta_{kj} \to 0$ ($k \to \infty, k \in \Delta_j$). If $J = \{j_1, \ldots, j_p\}$, then we can assume $\Delta_{j_p} \subset \cdots \subset \Delta_{j_1}$, so that $\lim \eta_{kj} = 0$ ($k \to \infty, k \in \Delta_{j_p}$) for all $j \in J$, hence (11). □

Now we apply this result to the problem (DC) when D is a polytope contained in a rectangle $[c, d] = \{x \in R^n : c \leqq x \leqq d\}$,

$$C = \{x \in R^n : f(x) \geqq \gamma), \ f(x) = \sum_{j=1}^{n} f_j(x_j) ,$$

and each $f_j(t)$ ($j = 1, \ldots, n$) is a concave function of one variable in the interval $[c_j, d_j]$.

For any subrectangle $M = [r, s] \subset [c, d]$, it is easily seen that the affine function $\phi_M(x)$ that agrees with $f(x)$ at the vertices of M is an underestimator of $f(x)$ on M:

$\phi_M(x) \leqq f(x) \ \forall x \in M$, hence

$$\min\{f(x) : x \in M \cap D\} \geqq \min\{\phi_M(x) : x \in M \cap D\} .$$

With this in mind, a branch and bound algorithm for solving the problem under consideration can be outlined as follows.

1) Let $\mathscr{P}_0 = \mathscr{M}_0 = \{[c, d]\}$. Set $k = 0$.

2) For each $M \in \mathscr{P}_k$, $M = [r, s]$ solve the linear program

$$\text{LP}(M) \quad \min \phi_M(x) \text{ s.t. } x \in M \cap D ,$$

obtaining the optimal value $\beta(M)$ and an optimal solution $\omega(M)$ of it. If for some M, $f(\omega(M)) < \gamma$, then terminate ($\omega(M) \in D \backslash C$). Otherwise, go to 3).

3) In $\mathscr{M}_k$ delete all M with $\beta(M) \geqq \gamma$. Let $\mathscr{R}_k$ be the collection of remaining rectangles. If $\mathscr{R}_k = \emptyset$ then terminate: $D \subset C$. Otherwise, go to 4).

4) Select $M_k \in \operatorname{argmin}\{\beta(M) : M \in \mathscr{R}_k\}$. Let $\omega(M_k) = \omega^k$, $\phi_k(x) = \phi_{M_k}(x) = \Sigma\phi_{kj}(x_j)$. Select an index set $J_k \subset \{1, \ldots, n\}$ containing an element j_k satisfying

$$j_k \in \underset{j}{\operatorname{argmax}} \, |f_j(\omega_j^k) - \phi_{kj}(\omega_j^k)| . \tag{12}$$

Subdivide M_k via (ω^k, J_k), obtaining a partition $\mathscr{P}_{k+1}$ of M_k.

5) In $\mathcal{R}_k$ replace M_k by $\mathcal{P}_{k+1}$. Let $\mathcal{M}_{k+1}$ be the resulting collection. Set $k \leftarrow k+1$ and return to 1).

THEOREM 5. *For any infinite nested sequence* $\{M_k, k \in \Gamma\}$ *generated by the algorithm we have*

$$\underline{\lim}\, |f(\omega^k) - \phi_k(\omega^k)| = 0 \ (k \to \infty, k \in \Gamma)\,. \tag{13}$$

Proof. By Theorem 4, without loss of generality we can assume that

$$\eta_{k1} = \omega_1^k - r_1^k \to 0 \quad (k \to \infty, k \in \Gamma); \quad 1 \in \operatorname*{argmax}_j |f_j(\omega_j^k) - \phi_{kj}(\omega_j^k)|. \tag{14}$$

From the continuity of $f_1(x)$ we have $|f_1(\omega_1^k) - f_1(r_1^k)| \to 0$. But $\omega_1^k = \alpha_{k1}s_1^k + (1 - \alpha_{k1})r_1^k$ with $\alpha_{k1} = \eta_{k1}/(s_1^k - r_1^k)$, hence $\phi_{k1}(\omega_1^k) = \alpha_{k1}\phi_{k1}(s_1^k) + (1-\alpha_{k1})\phi_{k1}(r_1^k) = \alpha_{k1} f_1(s_1^k) + (1-\alpha_{k1})f_1(r_1^k)$. Therefore, $|\phi_{k1}(\omega_1^k) - f_1(r_1^k)| = \alpha_{k1}|f_1(s_1^k) - f_1(r_1^k)| \to 0$ because either $s_1^k - r_1^k \to 0$ or $\alpha_{k1} \to 0$. Consequently,

$$|f_1(\omega_1^k) - \phi_{k1}(\omega_1^k)| \leqq |f_1(\omega_1^k) - f_1(r_1^k)| + |f_1(r_1^k) - \phi_{k1}(\omega_1^k)| \to 0\,.$$

Since $j_k = 1$ it then follows from (12) that

$$|f_j(\omega_j^k) - \phi_{kj}(\omega_j^k)| \to 0 \ \forall j = 1, \ldots, n$$

and this implies (13). □

THEOREM 6. *If* $D \backslash C$ *is nonempty then the above algorithm finds a point of* $D \backslash C$ *after finitely many iterations.*

Proof. Suppose that the algorithm is infinite. Then it generates an infinite nested sequence $\{M_k, k \in \Gamma\}$. By Theorem 5, we have (13). Since $f(\omega^k) \geqq \gamma$, while $\phi_k(\omega^k) = \beta(M_k)$, it follows that

$$\lim \beta(M_k) = \gamma(k \to \infty)\,. \tag{15}$$

(the monotonicity of the bounding: $\beta(M_k) \geqq \beta(M_h)\ \forall k > h$ can easily be derived from the concavity of $f(x)$). Now, if any point $x \in D$ belongs to a rectangle M which is deleted at some iteration k then $f(x) \geqq \beta(M) \geqq \gamma$. On the other hand, if at every iteration k, x belongs to some cone M in $\mathcal{R}_k$, then $f(x) \geqq \beta(M) \geqq \beta(M_k)$, hence again $f(x) \geqq \gamma$, by letting $k \to \infty$. Thus $D \subset C$, completing the proof. □

REMARKS. (i) Usually, the set J_k is chosen so that $J_k \subset \{j : f_j(\omega^k) - \phi_{kj}(\omega^k) > 0\}$. The above subdivision with $J_k = \{j_k\}$ was proposed in [1]. Of course, the possibility of taking a J_k larger than $\{j_k\}$ adds more flexibility to the method.

(ii) Sometimes we may be interested in having a stronger condition than (11), namely:

$$\delta_{kj_k} = s_{j_k}^k - r_{j_k}^k \to 0 \ (k \to \infty,\ k \in \Delta)$$

(exhaustiveness). It is not hard to see that the following choice of the subdivision point w^k will ensure this property ($N \geqq 1$ and $\eta \in (0,1)$ are user supplied parameters; $\tau(M)$ is the generation index of M):

(*) If $\tau(M)$ is a multiple of N and $\max\{|\omega^k_{j_k} - r^k_{j_k}|, |\omega^k_{j_k} - s^k_{j_k}|\} > \eta|s^k_{j_k} - r^k_{j_k}|$ then choose w^k so that $w^k_j = \omega^k_j$ for $j \neq j_k$ and $w^k_{j_k} = \frac{1}{2}(s^k_{j_k} + r^k_{j_k})$; otherwise, choose $w^k = \omega^k$.

Indeed, let $\{M_k, k \in \Gamma\}$ be any infinite nested sequence generated by this rule and consider any infinite sequence $\Delta \subset \Gamma$. Without loss of generality we can assume that $j_k = j$ (constant) for all $k \in \Delta$. Let Δ_1 denote the sequence of all $k \in \Delta$ for which the first alternative mentioned in the rule occurs. Since each time this alternative occurs the length of the segment $[r^k_j, s^k_j]$ decreases by a factor of η, it is clear that this length tends to 0 as $k \to \infty$ ($k \in \Delta$), provided Δ_1 is infinite. But if Δ_1 is finite, i.e. if there is k_1 such that the first alternative never occurs for $k \geqq k_1$ ($k \in \Delta$), then $\max\{|w^k_j - r^k_j|, |w^k_j - s^k_j|\} \leqq \eta|s^k_j - r^k_j|$ for infinitely many $k \geqq k_1$ ($k \in \Delta$), hence again $s^k_j - r^k_j \to 0$.

(iii) When each $f_j(x_j)$ is concave quadratic, it can be proved that there exist constants α_j ($j = 1, \dots, n$) satisfying $|f_j(\omega^k_j) - \phi_{kj}(\omega^k_j)| \leqq \alpha_j(s^k_j - r^k_j)^2$ (see [6]). Based on this property, in [6] an exhaustive subdivision was proposed such that $J_k = \{j_k\}$,

$$j_k \in \operatorname*{argmax}_j \alpha_j(s^k_j - r^k_j)^2\,,$$

and $w^k_{j_k} = \frac{1}{2}(s^k_{j_k} + r^k_{j_k})$. The drawback of this subdivision is its nonadaptive character, in contrast with the subdivision (*) which basically depends on ω^k. From the above development it appears that for the type of algorithms discussed in [6] the most efficient subdivision should be the ω-subdivision defined by (12). This remark can be illustrated by the following example (cf. [6]):

$$\begin{aligned}&\text{minimize } f(x) = -\tfrac{1}{2}(2x_1^2 + 8x_2^2) \text{ subject to}\\ &x_1 + x_2 \leqq 10,\ x_1 + 5x_2 \leqq 22,\ -3x_1 + 2x_2 \leqq 2\,,\\ &-x_1 - 4x_2 \leqq -4,\ x_1 - 2x_2 \leqq 4\,.\end{aligned}$$

With the subdivision method used in [6] and starting from $M_0 = \{x: 0 \leqq x_1 \leqq 8, 0 \leqq x_2 \leqq 4\}$, $x^0 = (8,2)$, $f(x^0) = -80$ seven iterations are needed to identify the global optimal solution $x_1 = 7$, $x_2 = 3$, whereas with the ω-subdivision only two iterations would suffice.

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